Lesson 19 - Differential Equations - Separation of Variables

I Definitions
II. Solving seperable Differential Equations (D.E.'s)
III. Examples
IV. Word Problems (Time Permiting)

I. Definitions

(D) A differential equation (D.E.) is an equation that relates a function and its derivatives.

A Solution of a D.E. is a function that satisfies the D.E.

y is a function of x

EX) Show that $y = e^{4x}$ is a solution to the D.E.

$$\underbrace{y + 2y' + y''}_{\text{LHS}} = \underbrace{25e^{4x}}_{\text{RHS}} \qquad (\star)$$

$y = e^{4x}$
$y' = 4e^{4x}$
$y'' = 16e^{4x}$

$$\begin{aligned} \text{LHS} = y + 2y' + y'' &= e^{4x} + 2(4e^{4x}) + 16e^{4x} \\ &= e^{4x} + 8e^{4x} + 16e^{4x} \\ &= 25e^{4x} \\ &= \text{RHS} \end{aligned}$$

So $y = e^{4x}$ is a solution to $(\star)$.

(D) A separable differential equation is a 1st-order (no 2nd derivatives or higher derivatives) D.E. that can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

multiplication

Ex $\frac{dy}{dx} = (x^2+2)(y+1)$ is separable

$\frac{dy}{dx} - xy^3 = 0$ is separable

$$\downarrow$$

$$\frac{dy}{dx} = xy^3$$

$\frac{dy}{dx} = x + y$ is <u>NOT</u> separable

$$\frac{dy}{dx} = e^{x+3y}$$

$$\downarrow$$

$\frac{dy}{dx} = e^{x}e^{3y}$ is separable

II. Solving Separable D.E.'s

$$\frac{dy}{dx} = f(y)g(x)$$

For $f(y) \neq 0$

$$\int \frac{1}{f(y)}\,dy = \int g(x)\,dx$$

<u>Notes</u>: (1) Make sure to add a constant at the end

(2) Solve for y to find our function.

III. Examples

Ex

$$\frac{dy}{dx} = x^2 y^2$$

$$\int \frac{1}{y^2}\,dy = \int x^2\,dx$$

$$\int y^{-2}\,dy = \int x^2\,dx$$

$$\frac{y^{-1}}{-1} = \frac{x^3}{3} + C$$

Now solve for y

$$\frac{-1}{y} = \frac{x^3}{3} + C$$

$$\frac{1}{y} = \frac{-x^3}{3} - C$$

$$y = \frac{1}{\frac{-x^3}{3} - C}\left(\frac{3}{3}\right)$$

$$y = \frac{3}{-x^3 \underbrace{-3C}_{\hat{C}}} = \frac{3}{-x^3 + \hat{C}}$$

LON-CAPA Ans $\quad y = \dfrac{3}{-x^3 + C}$

This is a general solution to the D.E.
It works for any value of the constant C.

$\boxed{Ex}$ Find a function $A(t)$ if $A > 50$ for all values of $t > 0$ and $A(0) = 80$

$$\frac{dA}{dt} = 10(A - 50)$$

initial value problem - IVP

$$\int \frac{dA}{(A-50)} = \int 10\, dt$$

$$\underbrace{\ln|A-50|}_{\substack{A>50 \Rightarrow \\ A-50>0 \\ \Rightarrow |A-50| = A-50}} = 10t + C$$

$$\ln(A-50) = 10t + C$$

Solve for A:

$$e^{\ln(A-50)} = e^{10t+C}$$

$$A - 50 = e^{10t} e^{C} \quad \text{constant}$$

$$A - 50 = ke^{10t}$$

$$A = 50 + ke^{10t}$$

$A(0) = 80$

$$80 = 50 + ke^{10(0)}$$

$$80 = 50 + k \cdot 1$$

$$k = 30$$

Ans: $A(t) = 50 + 30e^{10t}$.

IV. Word Problems

① A quantity y grows at a rate proportional to z if

$$\frac{dy}{dt} = kz$$

for some constant k.

A quantity y grows at a rate inversely proportional to z if

$$\frac{dy}{dt} = \frac{k}{z}$$

for some constant k.

NOTE: z could be y.

EX) A bacteria population grows at a rate proportional to its population. If the population is 50,000 @ $t=0$ and 100,000 @ $t=10$, find the population as a function of time.

y = population @ time t

$y > 0$

$$\frac{dy}{dt} = ky$$

$$\int \frac{dy}{y} = \int k\,dt$$

$$\ln(y) = kt + C$$

$$y = e^{kt} e^{c} \quad A$$

$$y = Ae^{kt}$$

$y(0) = 50,000$

$$50,000 = A\underbrace{e^{k\cdot 0}}_{1}$$

$$A = 50,000$$

$$y = 50,000\,e^{kt}$$

$y(10) = 100,000$

$$100,000 = 50,000\,e^{k\cancel{t}\,10}$$

$$2 = e^{k\cancel{t}\,10}$$

$$\ln(2) = 10k$$

$$k = \ln(2)/10$$

$$y(t) = 50,000\,e^{\frac{\ln(2)}{10}\cdot t}$$